Deformations of N = 2 superconformal algebra and supersymmetric two-component

Camassa–Holm equation

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Abstract
This paper is concerned with a link between central extensions of $N = 2$ superconformal algebra and a supersymmetric two-component generalization of the Camassa–Holm equation. Deformations of superconformal algebra give rise to two compatible bracket structures. One of the bracket structures is derived from the central extension and admits a momentum operator which agrees with the Sobolev norm of a co-adjoint orbit element. The momentum operator induces, via Lenard relations, a chain of conserved Hamiltonians of the resulting supersymmetric Camassa–Holm hierarchy.

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1. Introduction

The last few years have greatly enhanced our understanding of bi-Hamiltonian structures. Bi-Hamiltonian structure is frequently viewed as one of the key features of integrability and has recently been adopted as the basis for the classification programme (see e.g. [1–10]).

In recent publications [14, 16], a study of deformations of bi-Hamiltonian structures of hydrodynamic-type led to a new class of bi-Hamiltonian structure associated with a two-component generalization of the Camassa–Holm equation. In this construction the second bracket component of the bi-Hamiltonian structure contained a centreless Virasoro algebra

$$\{U(x), U(y)\}_2 = \delta'(x - y)(U(x) + U(y)) = 2U(x)\delta'(x - y) + U'(x)\delta(x - y) \tag{1}$$

coupled to a spin-one current via

$$\{U(x), v(y)\}_2 = \delta'(x - y)(v(x) + (s - 1)v(y)), \qquad s = 1. \tag{2}$$

Two totally different hierarchies emerged from deformations of a bi-Hamiltonian system depending on whether the first or second bracket structure has being deformed by the addition

of an appropriate central element. Deformation of the second bracket led to a standard AKNS hierarchy, while deformation of the first bracket gave rise to a new result: the two-component Camassa–Holm hierarchy. The Hamiltonians of the bi-Hamiltonian structure were shown to split into two chains: one chain of the positive order inducing the AKNS hierarchy and another of the negative order inducing the two-component Camassa–Holm hierarchy [15].

Understanding of central extensions of a Poisson structure is crucial for the analysis of deformations. Let us now consider a general Poisson bracket with a co-cycle and link it to the question of compatibility so essential in the theory of integrable systems. Generally, for a given Poisson bracket structure a co-cycle of this bracket provides an additional compatible bracket structure [19]. A trivial co-cycle routinely agrees with the conventional first bracket structure. For example, shifting $v \to v + c$ in (2) leads to a mixture of the second bracket with the first bracket of the type $\{U(x), v(y)\}_1 = \delta'(x - y)$. A non-trivial co-cycle can serve as a deformation of the first or, equally well, second structure. Algebraic properties of central elements ensure compatibility of all three Poisson bracket structures, meaning compatibility of trivial co-cycle, non-trivial co-cycle and the original bracket structure. In the setting of equation (2), the non-trivial central element is proportional to $\delta''(x - y)$ and it can be added to either (2) or the first bracket $\{U(x), v(y)\}_1 = \delta'(x - y)$ (see equation (25)) without violating the compatibility of the underlying bi-Hamiltonian structure. The above arguments constitute an algebraic reasoning behind a tri-Hamiltonian formalism introduced in [20].

In this paper, we will apply the above observations to the $N = 2$ superconformal algebra with a goal of building generalization of the two-component Camassa–Holm model which is invariant under supersymmetric transformations. This approach will not be based on $N = 2$ superspace formalism, like the one used in [21], but instead it will only employ the bi-Hamiltonian method.

We will apply the co-adjoint orbit method to the $N = 2$ superconformal model in order to develop a formalism, which derives equations of motion of the $N = 2$ supersymmetric Camassa–Holm model as Euler–Arnold equations. Misiolek [22] showed that the Camassa–Holm equation can be characterized as a geodesic flow on the Bott–Virasoro group. Guha and Olver [23] obtained a fermionic (but not supersymmetric) extension of the Camassa–Holm equation as a geodesic flow. Here we will fill this gap by providing a Sobolev $H^{(1)}$ inner product which is compatible with a supersymmetric structure inherent in the $N = 2$ superconformal algebra, and we will show that in this setting the supersymmetric Camassa–Holm equation follows as the Euler–Arnold equation.

Our formalism provides a direct guide on how to extend the conventional L_2 inner product to its deformed version given by the Sobolev $H^{(1)}$ inner product. In all models, we consider that the basic algebra defining the model is inducing the second bracket structure. The compatible first bracket structure is then obtained by a central extension containing the deformation parameter in front of the non-trivial co-cycle. In this setup the Sobolev $H^{(1)}$ inner product is derived by demanding that the norm of the generic algebra element is a momentum of the first bracket structure. This norm is then used to produce a chain of Hamiltonians defining the hierarchy via Lenard relations.

This paper is organized as follows. In section 2, we review the construction of the two-component Camassa–Holm equation within the co-adjoint orbit formalism emphasizing a general link between deformation of the first bracket and the Sobolev norm defining its momentum operator. In section 3, we construct a bi-Hamiltonian structure based on the $N = 2$ superconformal algebra with deformation contained in the first bracket structure. The method developed in section 2 leads to the momentum operator defining the generalized two-component Camassa–Holm hierarchy invariant under the $N = 2$ supersymmetry transformations. In section 4, we derive the $N = 2$ supersymmetric Camassa–Holm equations

as Euler-Arnold equations. In section 5, we show that the invariance under $N = 2$ supersymmetry transformations extends to the $N = 2$ Camassa–Holm model expressed in the hodographic variables when combined with the redefinition of the supersymmetry transformations to ensure a closure of the $N = 2$ algebra in new variables. We conclude the paper by presenting a brief outlook in section 6.

2. The two-component Camassa–Holm model

The semi-direct product $\mathcal{G}$ of the centreless Virasoro (Witt) and spin-one current algebra is given by

$$[(f, a), (g, b)] = (fg' - f'g, fb' - a'g). \tag{3}$$

The brackets (1) and (2) can then be derived using the Poisson bracket structure on a co-adjoint orbit $\mathcal{G}^*$ induced by algebra commutation relations via

$$\{F, G\}_P(\mu) = \left\langle \mu \left| \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu} \right] \right. \right\rangle, \tag{4}$$

where F, G are real functions on $\mathcal{G}^*$, μ is in $\mathcal{G}^*$ and $\langle \cdot | \cdot \rangle$ is an inner product.

Choosing F as $F = \langle (U, v) | (f, a) \rangle$, G as $G = \langle (U, v) | (g, b) \rangle$ and $\mu = (U, v)$ in definition (4) yields

$$\{\langle (U, v) | (f, a) \rangle, \langle (U, v) | (g, b) \rangle\}_P = \langle (U, v) | [(f, a), (g, b)] \rangle$$
$$= \langle (U, v) | (fg' - f'g, fb' - a'g) \rangle. \tag{5}$$

If $\langle \cdot | \cdot \rangle$ is taken to be the L_2 inner product

$$\langle (U, v) | (f, a) \rangle_{L_2} = \int [Uf + va] \, dx, \tag{6}$$

then $\{\cdot, \cdot\}_P$ in relation (5) reproduces a structure given in (1) and (2) together with

$$\{v(x), v(y)\}_2 = 0.$$

The Lie algebra (3) is compatible with a central extension

$$[(f, a, c_1, c_2), (g, b, c_1, c_2)] = (0, 0, \omega(a, b), \omega(f, g)), \tag{7}$$

using notation where c_1 and c_2 represent two additional central directions and

$$\omega(a, b) = \frac{1}{2} \langle (0, a'), (0, b) \rangle = \frac{1}{2} \int a'b \, dx,$$
$$\omega(f, g) = \frac{1}{2} \langle (f', 0), (g, 0) \rangle = \frac{1}{2} \int (f'g + \beta^2 f''g') \, dx.$$

The pairing (6) extends in a straightforward way to incorporate two extra central elements and implies via relation (5) the following new bracket structure:

$$\left\{ \langle (U, v) | (f, a) \rangle_{L_2}, \langle (U, v) | (g, b) \rangle_{L_2} \right\}_1 = \langle (U, v, c_1, c_2) | (0, 0, \omega(a, b), \omega(f, g)) \rangle_{L_2}$$
$$= \frac{1}{2} \int [c_1(f'g + \beta^2 f''g') + c_2(a'b)] \, dx. \tag{8}$$

The above bracket is compatible with a second bracket $\{\cdot, \cdot\}_2$ for all values of c_1, c_2. For $c_1 = c_2 = 1$, the above construction admits the following local first Poisson structure:

$$\{U(x), U(y)\}_1 = \frac{1}{2} \left(1 - \beta^2 \partial_x^2 \right) \delta'(x - y)$$
$$\{v(x), U(y)\}_1 = 0 \tag{9}$$
$$\{v(x), v(y)\}_1 = \frac{1}{2} \delta'(x - y).$$

The existence of two compatible bracket structures signals integrability and implies the presence of infinitely many conserved Hamiltonian structures. To construct these Hamiltonians, we first search for the object which acts as a gradient with respect to the first bracket $\{\cdot, \cdot\}_1$ from (9). We first investigate the L_2 norm of (U, v):

$$\langle (U, v)|(U, v)\rangle_{L_2} = \int (U^2 + v^2)\, dx.$$

For $h = (1/2)\langle (U, v)|(U, v)\rangle_{L_2}$, we find

$$\{h, U(x)\}_1 = \tfrac{1}{2}\big(1 - \beta^2 \partial_x^2\big) U'(x), \qquad \{h, v(x)\}_1 = \tfrac{1}{2} v'(x).$$

By redefining h as follows:

$$h \to \bar{h} = \frac{1}{2} \int \big(U\big(1 - \beta^2 \partial_x^2\big)^{-1} U + v^2\big)\, dx,$$

we now obtain the desired relations

$$\{\bar{h}, U(x)\}_1 = \tfrac{1}{2} U'(x), \qquad \{h, v(x)\}_1 = \tfrac{1}{2} v'(x),$$

but for the price of introducing a non-locality in the definition of $\bar{h}$. To remove this non-locality we need to redefine variable U to

$$U \longrightarrow m = U - \beta^2 U''.$$

Replacing U by m in $\bar{h}$ yields

$$\begin{aligned}
\bar{h} \longrightarrow H &= \frac{1}{2} \int \big(m\big(1 - \beta^2 \partial_x^2\big)^{-1} m + v^2\big)\, dx \\
&= \frac{1}{2} \int (Um + v^2)\, dx = \frac{1}{2} \int (U^2 + \beta^2 (U')^2 + v^2)\, dx,
\end{aligned}$$

where we assumed periodic boundary conditions. The above Hamiltonian H appears as a norm of (U, v):

$$H = \tfrac{1}{2}\langle (U, v)|(U, v)\rangle \tag{10}$$

with respect to the $H^{(1)}$ inner product

$$\langle (U, v)|(f, a)\rangle = \int [Uf + \beta^2 U' f' + va]\, dx. \tag{11}$$

When the bilinear form $\langle \cdot | \cdot \rangle$ in definition (4) is given by the above $H^{(1)}$ inner product, then $\{\cdot, \cdot\}_P$ from relation (5) reproduces a familiar second bracket structure:

$$\begin{aligned}
\{m(x), m(y)\}_2 &= 2m(x)\delta'(x - y) + m'(x)\delta(x - y) \\
\{v(x), m(y)\}_2 &= v(x)\delta'(x - y) + v'(x)\delta(x - y) \\
\{v(x), v(y)\}_2 &= 0.
\end{aligned} \tag{12}$$

Similarly, replacing the L_2 inner product by the H^1 inner product in relation (8) yields for the first bracket structure

$$\begin{aligned}
\{m(x), m(y)\}_1 &= \tfrac{1}{2}\big(1 - \beta^2 \partial_x^2\big)\delta'(x - y) \\
\{v(x), m(y)\}_1 &= 0 \\
\{v(x), v(y)\}_1 &= \tfrac{1}{2}\delta'(x - y).
\end{aligned} \tag{13}$$

In the remaining part of this section we will use the $H^{(1)}$ inner product as defined in (11). Thus, the canonical bracket structure $\{\cdot, \cdot\}_P$ agrees with that given by (12).

The co-adjoint action is obtained from

$$\langle \mathrm{ad}^*_{(f,a)}(g, b) | (h, c) \rangle = \langle (g, b) | [(f, a), (h, c)] \rangle$$

$$= \int [g(fh' - f'h) + \beta^2 g'(fh' - f'h)' + b(fc' - a'h)] \, \mathrm{d}x. \qquad (14)$$

For brevity, we introduce the following notation:

$$\mathrm{ad}^*_{(f,a)}(g, b) = (B_1, B_0).$$

Then the left-hand side of equation (14) equals

$$\int (B_1 h + \beta^2 B_1' h' + B_0 c) \, \mathrm{d}x = \int \left(h(1 - \beta^2 \partial_x^2) B_1 + B_0 c \right) \mathrm{d}x.$$

Comparing with the right-hand side of equation (14), we find

$$(1 - \beta^2 \partial_x^2) B_1 = -f(1 - \beta^2 \partial_x^2)g' - 2f'(1 - \beta^2 \partial_x^2)g - ba' \qquad B_0 = -(bf)'. \qquad (15)$$

Defining the Euler equations as

$$\frac{\partial}{\partial t}(U, v) = -\mathrm{ad}^*_{(U,v)}(U, v), \qquad (16)$$

we find from equation (15) setting $f = g = U, a = b = v$ that

$$\frac{\partial}{\partial t}(U, v) = -\left((1 - \beta^2 \partial_x^2)^{-1}\left[-U(1 - \beta^2 \partial_x^2)U' - 2U'(1 - \beta^2 \partial_x^2)U - vv'\right], -(vU)'\right) \qquad (17)$$

from which we derive the two-component Camassa–Holm equations

$$\frac{\partial}{\partial t}(1 - \beta^2 \partial_x^2)U = 3UU' - 2\beta^2 U'U'' - \beta^2 UU''' + vv'$$
$$\frac{\partial}{\partial t}v = (vU)'. \qquad (18)$$

With the Hamiltonian H from definition (10) and the bracket $\{\cdot, \cdot\}_2$ from equation (12), the Euler equations are derived as Hamiltonian equations through

$$\frac{\partial}{\partial t}(m, v) = (\{m(x), H\}_2, \{v(x), H\}_2) = (2mU' + m'U + vv', (vU)'). \qquad (19)$$

Note, that the last part of equations in (13) can also be written as

$$\{v^2(x), v^2(y)\}_1 = 2v^2(x)\delta'(x - y) + (v^2)'(x)\delta(x - y). \qquad (20)$$

With the above in mind, we can rewrite two compatible bracket structures $\{\cdot, \cdot\}_i, i = 1, 2$ in the matrix form as follows:

$$\begin{pmatrix} \{m(x), m(y)\}_i & \{m(x), v^2(y)\}_i \\ \{v^2(x), m(y)\}_i & \{v^2(x), v^2(y)\}_i \end{pmatrix} = P_i \delta(x - y), \qquad i = 1, 2. \qquad (21)$$

In the above basis, we obtain

$$P_2 = \begin{pmatrix} m\partial + \partial m & v^2\partial + \partial v^2 \\ v^2\partial + \partial v^2 & 0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} \frac{1}{2}(1 - \beta^2\partial^2)\partial & 0 \\ 0 & v^2\partial + \partial v^2 \end{pmatrix}, \qquad (22)$$

which give rise to the Hamiltonian operators. Remarkably, one now can define a third local compatible bracket structure through the formula $P_0 = P_1 P_2^{-1} P_1$. P_0 is explicitly given by

$$P_0 = \begin{pmatrix} 0 & \frac{1}{2}(1 - \beta^2\partial^2)\partial \\ \frac{1}{2}(1 - \beta^2\partial^2)\partial & -m\partial - \partial m \end{pmatrix}, \qquad (23)$$

or

$$\{m(x), m(y)\}_0 = 0$$
$$\{v^2(x), m(y)\}_0 = \tfrac{1}{2}\left(1 - \beta^2 \partial_x^2\right)\delta'(x - y) \qquad (24)$$
$$\{v^2(x), v^2(y)\}_0 = -2m(x)\delta'(x - y) - m'(x)\delta(x - y).$$

Defining the recursion operator $R = P_1 P_0^{-1}$, we can write the recurrence relation $P_2 = R P_1$, which connects the first and second bracket structures.

It is easy to see that the Hamiltonian H from equation (10) is a Casimir for the bracket structure $\{\cdot, \cdot\}_0$ given by (24). We also note that H acts as a gradient operator, $\{X(x), H\}_1 = X'(x)/2$ for $X = m, v$, for the first bracket structure, $\{\cdot, \cdot\}_1$, from (13).

For new variables w_1, w_2 connected to m, v via

$$m = \frac{1}{2}(w_1 + \beta w_1'), \qquad w_2 = v^2 + \frac{w_1^2}{4},$$

the two lowest Poisson brackets

$$\{w_1(x), w_1(y)\}_1 = 2\delta'(x - y),$$
$$\{w_1(x), w_2(y)\}_1 = w_1(x)\delta'(x - y) + w_1'(x)\delta(x - y),$$
$$\{w_2(x), w_2(y)\}_1 = 2w_2(x)\delta'(x - y) + w_2'(x)\delta(x - y), \qquad (25)$$
$$\{w_1(x), w_1(y)\}_0 = \{w_2(x), w_2(y)\}_0 = 0,$$
$$\{w_1(x), w_2(y)\}_0 = \delta'(x - y) - \beta\delta''(x - y)$$

form for $\beta = 0$ a standard bi-Hamiltonian structure which contains the second bracket structure from equations (1) and (2). For a non-zero value of the deformation parameter, $\beta \neq 0$, relations (25) describe a deformation of a bi-Hamiltonian structure with the deformed first Poisson structure [16]. This is that deformation of the bi-Hamiltonian structure which gave rise to the two-component Camassa–Holm equation in [14, 16]. In the following section, we will focus on the similar deformation of the larger $N = 2$ superconformal Poisson algebra.

3. The $N = 2$ superconformal algebra

A starting point is a general second bracket structure (see for instance [17] for $\epsilon = 1$ and $\kappa = 0$):

$$\{w_1(x), w_1(y)\}_2 = 2\epsilon\delta'(x - y),$$
$$\{w_1(x), w_2(y)\}_2 = w_1(x)\delta'(x - y) + w_1'(x)\delta(x - y) - \epsilon\delta''(x - y) + \kappa\delta'(x - y),$$
$$\{w_2(x), w_2(y)\}_2 = 2w_2(x)\delta'(x - y) + w_2'(x)\delta(x - y),$$
$$\{w_1(x), \xi_1(y)\}_2 = \xi_1(x)\delta(x - y),$$
$$\{w_1(x), \xi_2(y)\}_2 = -\xi_2(x)\delta(x - y), \qquad (26)$$
$$\{w_2(x), \xi_1(y)\}_2 = 2\xi_1(x)\delta'(x - y) + \xi_1'(x)\delta(x - y),$$
$$\{w_2(x), \xi_2(y)\}_2 = \xi_2(x)\delta'(x - y),$$
$$\{\xi_1(x), \xi_2(y)\}_2 = -\tfrac{1}{4}w_2(x)\delta(x - y) - \tfrac{1}{4}w_1(x)\delta'(x - y) - \tfrac{1}{4}\epsilon\delta''(x - y) - \tfrac{1}{4}\kappa\delta'(x - y).$$

It distinguishes itself from the second bracket from (1), (2) by the presence of additional fermionic variables ξ_i, $i = 1, 2$ and central elements. The terms with ϵ describe the deformation of a second bracket which is compatible with the Jacobi relations. The terms with a constant κ are trivial co-cycle terms which can easily be absorbed by a shift $w_1 \to w_1 - \kappa$.

Algebra (26) (with $\kappa = 0$) can easily be put in the form considered by Labelle and Mathieu [18] in terms of the following change of variables:

$$u_{\mathrm{LM}}(x) = 2w_2(x) - w_1'(x), \qquad w_{\mathrm{LM}}(x) = iw_1(x)$$
$$\xi_{\mathrm{LM1}}(x) = 2i(\xi_1(x) + \xi_2(x)), \qquad \xi_{\mathrm{LM2}}(x) = 2(\xi_1(x) - \xi_2(x)). \qquad (27)$$

The above second bracket is compatible with the first bracket structure

$$\{w_1(x), w_1(y)\}_1 = 2\beta\delta'(x - y)$$
$$\{w_1(x), w_2(y)\}_1 = \delta'(x - y) - \beta\delta''(x - y)$$
$$\{\xi_1(x), \xi_2(y)\}_1 = -\frac{1}{4}\delta'(x - y) - \frac{\beta}{4}\delta''(x - y),$$

(28)

where terms with a constant β describe the deformation of a first bracket which is compatible with the second bracket in (26).

Comparing the bracket structures in (26) and (28) we note that they have identical central elements for $\epsilon = \beta$ and $\kappa = 1$.

Next, we define the quantity

$$H = \int dx[w_1(1 + \beta\partial_x)^{-1}w_2 - \beta((1 + \beta\partial_x)^{-1}w_2)^2 + 4\xi_1(1 - \beta\partial_x)^{-1}\xi_2],$$

(29)

which acts as a gradient with respect to the first bracket $\{\cdot, \cdot\}_1$ from (28), meaning that

$$\{X(x), H\}_1 = X'(x) \qquad \text{for} \qquad X = w_i, \xi_i, \quad i = 1, 2.$$

The gradient operator H contains non-localities and as in section 2 we will absorb non-localities within new redefined variables. For this purpose, we now introduce variables $\phi_i, \eta_i, i = 1, 2$ such that they allow rewriting the first bracket (28) as a Heisenberg-like algebra :

$$\{w_i(x), \phi_i(y)\}_1 = 0, \qquad \{w_i(x), \phi_j(y)\}_1 = \delta'(x - y), \qquad i \neq j, \quad i, j = 1, 2 \qquad (30)$$

and

$$\{\xi_2(x), \eta_1(y)\}_1 = -\{\xi_1(x), \eta_2(y)\}_1 = \tfrac{1}{4}\delta'(x - y).$$

It is easy to verify that original variables $w_i, \xi_i, i = 1, 2$ must be related to $\phi_i, \eta_i, i = 1, 2$ through relations

$$w_1 = 2\beta\phi_2 + \phi_1 - \beta\phi_1', \qquad w_2 = \phi_2 + \beta\phi_2', \qquad \xi_1 = \eta_1 + \beta\eta_1', \qquad \xi_2 = \eta_2 - \beta\eta_2'.$$

Now, set

$$u = \tfrac{1}{2}\phi_1, \qquad v = \tfrac{1}{2}w_1$$

and define m as a linear combination

$$m = c_1 w_2 + c_2 w_1 + c_3 w_1'.$$

The coefficients c_1, c_2, c_3 will be chosen in such a way as to ensure that m and v are decoupled, with respect to the first bracket structure, meaning that the first bracket of m with v vanishes. In view of equation (30), this requirement implies that m should not contain ϕ_2 or in different words that m should be given by

$$m = c_1 \left(w_2 - \frac{1}{2\beta}w_1 - \frac{1}{2}w_1' \right).$$

In the following, we set $c_1 = -\beta$ and accordingly use the definition

$$m = u - \beta^2 u_{xx} = \tfrac{1}{2}(w_1 + \beta w_{1x}) - \beta w_2; \qquad \beta\phi_2 = v - u + \beta u_x.$$

Note that the first bracket (28) reads in terms of m, v and $\xi_i, \eta_i, i = 1, 2$ as

$$\{m(x), m(y)\}_1 = -\tfrac{1}{2}\beta(1 - \beta^2\partial_x^2)\delta'(x - y)$$
$$\{v(x), m(y)\}_1 = 0$$
$$\{v(x), v(y)\}_1 = \tfrac{1}{2}\beta\delta_x(x - y)$$
$$\{\xi_i(x), \eta_j(y)\}_1 = -\epsilon_{ij}\tfrac{1}{4}\delta'(x - y), \qquad i, j = 1, 2,$$

(31)

where $\epsilon_{12} = -\epsilon_{21} = 1$ and zero otherwise. Also, note that the first three equations agree with the first bracket structure in (13) up to an overall factor of $\pm\beta$. The gradient operator (29) becomes in this notation

$$H = \int \left[\phi_2 w_1 - \beta\phi_2^2 - 4\eta_2\xi_1\right] dx$$

$$= -\frac{1}{\beta} \int dx[um - v^2 - 4\beta\xi_1\eta_2] \tag{32}$$

and it holds that $\{X(x), H\}_1 = X'(x)$ for $X = m, v, \xi_i, i = 1, 2$. It is important to point out that H is invariant under the following $N = 2$ supersymmetry transformation:

$$\delta\xi_1 = \frac{\epsilon_1 - \epsilon_2}{2\beta}(m - (1 + \beta\partial_x)v), \qquad \delta\xi_2 = \frac{\epsilon_1 + \epsilon_2}{2\beta}(-m + (1 + \beta\partial_x)v)$$

$$\delta u = \epsilon_1(\eta_1 + \eta_2) + \epsilon_2(\eta_1 - \eta_2), \qquad \delta v = \epsilon_1(\xi_1 + \xi_2) + \epsilon_2(\xi_1 - \xi_2) \tag{33}$$

meaning that $\delta H = 0$. This invariance of H will later lead to the invariance of the whole integrable model to be defined below.

Also, below we will see that H will agree with the Sobolev norm appearing in the co-adjoint treatment of the model.

In terms of m and v and for $\kappa = \epsilon/\beta$, the second bracket (26) takes the following form:

$$\{m(x), m(y)\}_2 = -\beta(2m(x)\delta_x(x - y) + m_x(x)\delta(x - y)) - \frac{1}{2}\epsilon\left(1 - \beta^2\partial_x^2\right)\delta'(x - y)$$

$$\{v(x), m(y)\}_2 = -\beta(v(x)\delta_x(x - y) + v_x(x)\delta(x - y))$$

$$\{v(x), v(y)\}_2 = \frac{1}{2}\epsilon\delta_x(x - y)$$

$$\{v(x), \xi_1(y)\}_2 = \frac{1}{2}\xi_1(x)\delta(x - y),$$

$$\{v(x), \xi_2(y)\}_2 = -\frac{1}{2}\xi_2(x)\delta(x - y),$$

$$\{m(x), \xi_1(y)\}_2 = \frac{1}{2}\xi_1(x)\delta(x - y) - \frac{\beta}{2}(3\xi_1(x)\delta'(x - y) + \xi_1'(x)\delta(x - y)), \tag{34}$$

$$\{m(x), \xi_2(y)\}_2 = -\frac{1}{2}\xi_2(x)\delta(x - y) - \frac{\beta}{2}(3\xi_2(x)\delta'(x - y) + \xi_2'(x)\delta(x - y)),$$

$$\{\xi_1(x), \xi_2(y)\}_2 = \frac{m(x) - v(x)}{4\beta}\delta(x - y) - \frac{1}{4}(v'\delta(x - y) + 2v(x)\delta'(x - y))$$

$$- \frac{\epsilon}{\beta}\left(\frac{1}{4}\delta'(x - y) + \frac{\beta}{4}\delta''(x - y)\right).$$

We note that a co-cycle of the second bracket provides an additional compatible bracket structure. From now on, we set ϵ to zero (while maintaining $\beta \neq 0$ in (28)) and describe the bi-Hamiltonian structure consisting of brackets $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$, which gives rise to the Lenard relations

$$\{w_i(x), H_{-j}\}_2 = j\{w_i(x), H_{-j-1}\}_1, \qquad j = 1, 2, 3, \ldots. \tag{35}$$

The Lenard relations (35) can be used to recursively build a hierarchy of Hamiltonians. Starting with the gradient operator $H_{-2} = H$ from (32) we obtain from relations (35) an expression for $H_{-1} = \int w_2(x) dx$, which happens to be the Casimir of the first Poisson bracket. Similarly, we also find an expression for Hamiltonian H_{-3}:

$$H_{-3} = \frac{1}{2} \int \left[w_1\phi_1\phi_2 - \beta\phi_1\phi_2^2 - 4\beta\phi_2\eta_2\eta_1 - 4\phi_1(2\eta_2\eta_1 + \beta\eta_2\eta_1')\right] dx,$$

as an application of the Lenard relations (35) (for $\epsilon = 0$).

We will study the flow of the bi-Hamiltonian hierarchy defined by

$$\frac{\partial w_i}{\partial t} = \{w_i(x), H_{-3}\}_1 = \frac{1}{2}\{w_i(x), H_{-2}\}_2$$

$$\frac{\partial \xi_i}{\partial t} = \{\xi_i(x), H_{-3}\}_1 = \frac{1}{2}\{\xi_i(x), H_{-2}\}_2, \qquad i = 1, 2. \tag{36}$$

In this way we obtain

$$w_{1t} = \tfrac{1}{2}(w_1\phi_1 - 4\beta\eta_2\eta_1)_x$$
$$w_{2t} = \tfrac{1}{2}\left(2\phi_1\phi_2 + \beta\phi_2^2 + \beta\phi_1\phi_{2x} - 4(2\eta_2\eta_1 + \beta\eta_2\eta_{1x})\right)_x$$
$$\xi_{1t} = \tfrac{1}{2}(2\eta_1\phi_1 + \beta\eta_{1x}\phi_1 + \beta\eta_1\phi_2)_x$$
$$\xi_{2t} = \tfrac{1}{2}(\beta\eta_2\phi_2 + 2\eta_2\phi_1 - \beta(\eta_2\phi_1)_x)_x. \tag{37}$$

Equations of motion (37) become in terms of m, u and v:

$$m_t = 3uu_x - \beta^2 uu_{xxx} - 2\beta^2 u_x u_{xx} - vv_x + (\beta\eta_2\eta_1 + \beta\xi_2\eta_1 + \beta\eta_2\xi_1)_x$$
$$v_t = (vu + \beta\eta_1\eta_2)_x$$
$$\xi_{1t} = \tfrac{1}{2}(3\eta_1 u + 2\beta\eta_{1x}u + \eta_1(v + \beta u_x))_x$$
$$\xi_{2t} = \tfrac{1}{2}(3\eta_2 u - 2\beta\eta_{2x}u + \eta_2(v - \beta u_x))_x. \tag{38}$$

Let

$$\psi_1 = (\xi_1 - \xi_2)\sqrt{2}, \qquad \psi_2 = (\xi_1 + \xi_2)\sqrt{2}.$$

Then the fermionic part of (34) takes in terms of ψ_i, $i = 1, 2$, the following form:

$$\{v(x), \psi_1(y)\}_2 = \frac{1}{2}\psi_2(x)\delta(x - y),$$

$$\{v(x), \psi_2(y)\}_2 = \frac{1}{2}\psi_1(x)\delta(x - y),$$

$$\{m(x), \psi_1(y)\}_2 = \frac{1}{2}\psi_2(x)\delta(x - y) - \frac{\beta}{2}(\psi_1'(x)\delta(x - y) + 3\psi_1(x)\delta'(x - y)),$$

$$\{m(x), \psi_2(y)\}_2 = \frac{1}{2}\psi_1(x)\delta(x - y) - \frac{\beta}{2}(\psi_2'(x)\delta(x - y) + 3\psi_2(x)\delta'(x - y)), \tag{39}$$

$$\{\psi_1(x), \psi_1(y)\}_2 = -\{\psi_2(x), \psi_2(y)\}_2 = \frac{1}{\beta}(m(x) - v(x))\delta(x - y)$$

$$\{\psi_1(x), \psi_2(y)\}_2 = -2v(x)\delta'(x - y) - v'(x)\delta(x - y).$$

One can remove the deformation parameter β from the Poisson brackets (34) and (39) by defining

$$M = -\frac{1}{\beta}(m - v) \tag{40}$$

in terms of which the brackets involving m can be expressed as

$$\{M(x), M(y)\}_2 = 2M(x)\delta_x(x - y) + M_x(x)\delta(x - y)$$
$$\{v(x), M(y)\}_2 = v(x)\delta_x(x - y) + v_x(x)\delta(x - y)$$
$$\{M(x), \psi_i(y)\}_2 = \tfrac{1}{2}\psi_i'(x)\delta(x - y) + \tfrac{3}{2}\psi_i(x)\delta'(x - y), \qquad i = 1, 2 \tag{41}$$
$$\{\psi_1(x), \psi_1(y)\}_2 = -\{\psi_2(x), \psi_2(y)\}_2 = -M(x)\delta(x - y).$$

Thus the deformation parameter β has been effectively moved from the Poisson brackets to the redefined function M.

Introducing

$$\gamma_1 = \eta_1 - \eta_2, \qquad \gamma_2 = \eta_1 + \eta_2,$$

we can rewrite ψ_1, ψ_2 as

$$\psi_1 = (\gamma_1 + \beta\gamma_{2x})\sqrt{2}, \qquad \psi_2 = (\gamma_2 + \beta\gamma_{1x})\sqrt{2}.$$

In this notation, the equations of motion (38) become

$$
\begin{aligned}
&m_t = 3uu_x - \beta^2 uu_{xxx} - 2\beta^2 u_x u_{xx} - vv_x + \tfrac{1}{2}(3\beta\gamma_2\gamma_1 + \beta^2\gamma_2\gamma_{2x} - \beta^2\gamma_1\gamma_{1x})_x \\
&v_t = \left(vu + \tfrac{1}{2}\beta\gamma_1\gamma_2\right)_x \\
&(\gamma_1 + \beta\gamma_{2x})_t = \tfrac{1}{2}(3u\gamma_1 + 2\beta u\gamma_{2x} + v\gamma_1 + \beta u_x\gamma_2)_x \\
&(\gamma_2 + \beta\gamma_{1x})_t = \tfrac{1}{2}(3u\gamma_2 + 2\beta u\gamma_{1x} + v\gamma_2 + \beta u_x\gamma_1)_x.
\end{aligned}
\tag{42}
$$

These equations of motions can be cast in a manifestly supersymmetric form

$$[\beta(D_1 D_2\Phi) - \Phi]_t = \left[\beta\Phi(D_1 D_2\Phi) - \Phi^2 - \frac{\beta}{2}(D_2\Phi)(D_1\Phi)\right]_x, \tag{43}$$

where the covariant derivatives D_1, D_2 defined in terms of Grassmannian variables θ_1, θ_2 as

$$D_1 = \frac{\partial}{\partial\theta_1} - \theta_1\partial_x, \qquad D_2 = \frac{\partial}{\partial\theta_2} + \theta_2\partial_x \tag{44}$$

satisfy

$$D_1^2 = -\frac{\partial}{\partial x}, \qquad D_2^2 = \frac{\partial}{\partial x},$$

and where we assumed that $\theta_i\gamma_j = -\gamma_j\theta_i$ for $i, j = 1, 2$.

The superfield

$$\Phi = u + \theta_1\gamma_2 + \theta_2\gamma_1 + \theta_2\theta_1\frac{u - v}{\beta} \tag{45}$$

transforms under supersymmetry as

$$\delta\Phi = (\epsilon_1 Q_1 + \epsilon_2 Q_2)\Phi, \tag{46}$$

where the generators Q_1, Q_2 are

$$Q_1 = \frac{\partial}{\partial\theta_1} + \theta_1\partial_x, \qquad Q_2 = \frac{\partial}{\partial\theta_2} - \theta_2\partial_x. \tag{47}$$

The supersymmetry transformation (46) reads in components

$$
\begin{aligned}
&\delta\gamma_1 = \epsilon_1\frac{u - v}{\beta} + \epsilon_2 u_x && \delta\gamma_2 = -\epsilon_1 u_x - \epsilon_2\frac{u - v}{\beta} \\
&\delta u = \epsilon_1\gamma_2 + \epsilon_2\gamma_1 && \delta v = \epsilon_1(\gamma_2 + \beta\gamma_{1x}) + \epsilon_2(\gamma_1 + \beta\gamma_{2x}).
\end{aligned}
\tag{48}
$$

It is easy to explicitly verify that the equations of motion (42) are invariant under supersymmetry transformations (48).

Equation (43) agrees with the supersymmetric $N = 2$, $\alpha = 4$ Camassa–Holm equation obtained in [21] using the hereditary recursion operator defined on a superspace.

4. The co-adjoint orbit method for the $N = 2$ model

Here we will develop a formalism, which derives the equations of motion (38) as the Euler–Arnold equations. The algebra elements of the $N = 2$ superconformal algebra $\mathcal{G}$ will be denoted as $(f, a, \alpha_1, \alpha_2)$, (g, b, β_1, β_2) and so on. They satisfy the commutation relations

$$[(f, a, \alpha_1, \alpha_2), (g, b, \beta_1, \beta_2)] = \bigg(fg' - f'g - \alpha_2\beta_1 - \alpha_1\beta_2, \, fb' - a'g - \alpha_2\beta_1 - \alpha_1\beta_2$$

$$- \beta\alpha_2\beta_1' + \beta\alpha_2'\beta_1 + \beta\alpha_1\beta_2' - \beta\alpha_1'\beta_2, \, -\frac{1}{2\beta}a\beta_1 + \frac{1}{2\beta}b\alpha_1 + \frac{1}{2\beta}f\beta_1$$

$$- \frac{1}{2\beta}g\alpha_1 + f\beta_1' - \frac{1}{2}f'\beta_1 - g\alpha_1' + \frac{1}{2}g'\alpha_1, \, \frac{1}{2\beta}a\beta_2 - \frac{1}{2\beta}b\alpha_2 - \frac{1}{2\beta}f\beta_2$$

$$+ \frac{1}{2\beta}g\alpha_2 + f\beta_2' - \frac{1}{2}f'\beta_2 - g\alpha_2' + \frac{1}{2}g'\alpha_2 \bigg). \tag{49}$$

Its dual space will be denoted as $\mathcal{G}^*$. The typical element of $\mathcal{G}^*$ is denoted by (u, v, η_1, η_2) The pairing between $\mathcal{G}$ and $\mathcal{G}^*$ will be provided by the $H^{(1)}$ inner product defined as

$$\langle (u, v, \eta_1, \eta_2)|(f, a, \alpha_1, \alpha_2) \rangle = \int \bigg[\frac{va}{\beta} - \frac{1}{\beta}(uf + \beta^2 u' f') + 2(\eta_1 + \beta\eta_1')\alpha_2$$

$$- 2(\eta_2 - \beta\eta_2')\alpha_1 \bigg] \mathrm{d}x$$

$$= \int \bigg[\frac{va}{\beta} - \frac{1}{\beta}fm + 2\xi_1\alpha_2 - 2\xi_2\alpha_1 \bigg] \mathrm{d}x. \tag{50}$$

The co-adjoint action is obtained from

$$\big\langle \mathrm{ad}^*_{(f,a,\alpha_1,\alpha_2)}(u, v, \eta_1, \eta_2)|(g, b, \beta_1, \beta_2) \big\rangle = \langle (u, v, \eta_1, \eta_2)|[(f, a, \alpha_1, \alpha_2), (g, b, \beta_1, \beta_1)] \rangle. \tag{51}$$

Denote $\mathrm{ad}^*_{(f,a,\alpha_1,\alpha_2)}(u, v, \eta_1, \eta_2)$ by (B, B_0, B_1, B_2). Equation (51) yields

$$\big(1 - \beta^2\partial_x^2\big)B = -(fm)' - f'm + va' - (\xi_1\alpha_2 + \xi_2\alpha_1) + 2\beta(\xi_1\alpha_2' - \xi_2\alpha_1') + \beta(\xi_1\alpha_2 - \xi_2\alpha_1)'$$

$$B_0 = -(fv)' + \xi_1\alpha_2 + \xi_2\alpha_1$$

$$(1 + \beta\partial_x)B_1 = \frac{1}{2}\bigg(\frac{1}{\beta}\alpha_1 m - \alpha_1'v - (\alpha_1 v)' - \frac{1}{\beta}\alpha_1 v + \frac{1}{\beta}\xi_1 a - \frac{1}{\beta}\xi_1 f - \xi_1 f' - 2(\xi_1 f)' \bigg)$$

$$(1 - \beta\partial_x)B_2 = -\frac{1}{2}\bigg(\frac{1}{\beta}\alpha_2 m + \alpha_2'v + (\alpha_2 v)' - \frac{1}{\beta}\alpha_2 v + \frac{1}{\beta}\xi_2 a - \frac{1}{\beta}\xi_2 f + \xi_2 f' + 2(\xi_2 f)' \bigg).$$

Substituting (u, v, η_1, η_2) for $(f, a, \alpha_1, \alpha_2)$ in the above equations, we see that the equations of motion (38) are reproduced as the Euler–Arnold equations

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, v, \eta_1, \eta_2) = -\mathrm{ad}^*_{(u,v,\eta_1,\eta_2)}(u, v, \eta_1, \eta_2) \tag{52}$$

The above equation can also be derived using the Poisson bracket structure on a co-adjoint orbit $\mathcal{G}^*$ induced by algebra commutation relations via relation (4). For

$$F = \langle (u, v, \eta_1, \eta_2)|(f, a, \alpha_1, \alpha_2) \rangle,$$

$$G = \langle (u, v, \eta_1, \eta_2)|(g, b, \beta_1, \beta_2) \rangle, \tag{53}$$

$$\mu = (u, v, \eta_1, \eta_2),$$

we find from relation (4) the bracket structure which agrees with $\{\cdot, \cdot\}_2$ from (34).

After multiplying the right-hand side of equation (52) by $|(f, a, \alpha_1, \alpha_2)\rangle$ we obtain

$$-\langle \mathrm{ad}^*_{(u,v,\eta_1,\eta_2)}(u, v, \eta_1, \eta_2)|(f, a, \alpha_1, \alpha_2)\rangle = \langle (u, v, \eta_1, \eta_2)|\mathrm{ad}_{(f,a,\alpha_1,\alpha_2)}(u, v, \eta_1, \eta_2)\rangle.$$

In view of definition (4) and due to the above relation the Euler–Arnold equation can be cast in a form

$$\frac{\mathrm{d}}{\mathrm{d}t} F = \{F, H\}$$

for F given by definition (53) and for H being a half of the H^1-like norm of (u, v, η_1, η_2) given explicitly by

$$
\begin{aligned}
H &= \frac{1}{2}\langle (u, v, \eta_1, \eta_2)|(u, v, \eta_1, \eta_2)\rangle \\
&= \frac{1}{2}\int \left[\frac{v^2}{\beta} - \frac{1}{\beta}(u^2 + \beta^2 (u')^2) + 2(\eta_1 + \beta\eta_1')\eta_2 - 2(\eta_2 - \beta\eta_2')\eta_1 \right] \mathrm{d}x \\
&= \frac{1}{2} H_{-2},
\end{aligned}
\tag{54}
$$

with H_{-2} as defined in (32).

A central extension of algebra (49) is given by the following commutation relations:

$$
\begin{aligned}
&[(f, a, \alpha_1, \alpha_2, c, c_0, c_{12}), (g, b, \beta_1, \beta_2, c, c_0, c_{12})]_c \\
&= \frac{\epsilon}{2\beta^2}\left(0, 0, 0, 0, \int f(1 - \beta^2\partial_x^2)g', \int ab', 2\beta \int (1 - \beta\partial_x)\alpha_2'\beta_1 - (1 + \beta\partial_x)\alpha_1'\beta_2\right).
\end{aligned}
\tag{55}
$$

Via the extension of relation (4), the above central extension leads to the Poisson bracket structure

$$
\begin{aligned}
\{m(x), m(y)\}_c &= -\frac{1}{2}\epsilon\left(1 - \beta^2\partial_x^2\right)\delta_x(x - y) \\
\{v(x), m(y)\}_c &= 0 \\
\{v(x), v(y)\}_c &= \frac{1}{2}\epsilon\delta_x(x - y) \\
\{\xi_1(x), \xi_2(y)\}_c &= \frac{\epsilon}{\beta}\left(-\frac{1}{4}\delta'(x - y) - \frac{\beta}{4}\delta''(x - y)\right),
\end{aligned}
\tag{56}
$$

which agrees both with the first bracket structure in (31) and the central extension of the second bracket structure in (26).

Note that the quantity

$$H_{-1} = \frac{1}{\beta}\int (v - m) \, \mathrm{d}x$$

commutes with all variables with respect to the first bracket structures (56) or (31). With respect to the second bracket structure (26) H_{-1} acts as a gradient operator

$$\{X(x), H_{-1}\}_2 = X'(x), \qquad \text{for} \quad X = m, v, \xi_1, \xi_2. \tag{57}$$

We find from the above that

$$
\begin{aligned}
\{F, H_{-1}\}_2 &= \langle (u', v', \eta_1', \eta_2')|(f, a, \alpha_1, \alpha_2)\rangle \\
&= \int \left[\frac{v'a}{\beta} - \frac{1}{\beta}f(1 - \beta^2\partial_x^2)u' + 2(1 + \beta\partial_x)\eta_1'\alpha_2 - 2(1 - \beta\partial_x)\eta_2'\alpha_1 \right] \mathrm{d}x \\
&= \frac{\epsilon}{2\beta}\langle (u, v, \eta_1, \eta_2)|[(f, a, \alpha_1, \alpha_2, c, c_0, c_{12}), (u, v, \eta_1, \eta_2, c, c_0, c_{12})]_c\rangle.
\end{aligned}
$$

After setting $\epsilon = \beta$, we obtain that the above equation equals

$$\{F, H_{-1}\}_2 = \frac{1}{2}\left\langle (u, v', \eta_1', \eta_2') \left| \left[(f, a, \alpha_1, \alpha_2, c, c_0, c_{12}), \frac{\delta}{\delta\mu} |(u, v, \eta_1, \eta_2)|^2 \right]_c \right\rangle \right.$$
$$= \{F, H\}$$

for H from (54), F from (53) and with $\mu = (u, v, \eta_1, \eta_2)$.

Consider the first bracket structure given by (56) with $\epsilon = \beta$ or (31), then the Hamiltonian H from (54) serves as a momentum with respect to this bracket structure according to

$$\{X(x), H\}_1 = \{X(x), H_{-1}\}_2 = X'(x), \qquad \text{for} \quad X = m, v, \xi_1, \xi_2 \tag{58}$$

as follows from (57) and the Lenard relation proved above. This shows that the H^1-like norm is fully fixed by the requirement that the norm H serves as a momentum with respect to the first bracket structure defined by a central extension. Furthermore, applying the Lenard argument we find

$$\{X(x), H_{-1}\}_1 = \{X(x), H_0\}_2,$$

with H_0 being a constant by a power of fields counting argument. Thus, we have proved that H_{-1} is a Casimir of the first bracket structure.

5. Hodographic transformation and supersymmetry

It is customary to study the Camassa–Holm equation in terms of hodographic variables (see for instance [16]). The basic question which we will address in this section is how the change of variables to the hodographic variables affects supersymmetry described above. The difficulty one encounters here is the fact that the hodographic variables do not commute with the regular supersymmetry transformations. This obstacle makes it necessary to define a new set of extended supersymmetry transformations obeying the $N = 2$ supersymmetric algebra and commuting with hodographic variables. In the next step, we rewrite all equations of motion (42) as continuity equations in terms of the hodographic variables which leads to the proof of their invariance under the $N = 2$ supersymmetry transformations.

Here, we will study a hodographic change of variables based on a continuity equation, $v_t = \left(vu + \frac{1}{2}\beta\gamma_1\gamma_2\right)_x$, the second part of equations in (42). It follows from this continuity equation that a 1-form

$$\omega = v\,\mathrm{d}x + \left(vu + \tfrac{1}{2}\beta\gamma_1\gamma_2\right)\mathrm{d}t$$

is closed, meaning that $\mathrm{d}\omega = 0$ with $\mathrm{d} = \mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}t\frac{\partial}{\partial t}$. Thus, it is possible to rewrite ω as

$$\omega = \mathrm{d}y$$

for some scalar function y. Let s be such that $\mathrm{d}s = \mathrm{d}t$ and choose now a new coordinate system with coordinates y and s. Then in the new coordinate system an expression for the exterior derivative d becomes

$$\mathrm{d} = \mathrm{d}y\frac{\partial}{\partial y} + \mathrm{d}s\frac{\partial}{\partial s} = \left(v\,\mathrm{d}x + \left(vu + \frac{1}{2}\beta\gamma_1\gamma_2\right)\mathrm{d}t\right)\frac{\partial}{\partial y} + \mathrm{d}t\frac{\partial}{\partial s}.$$

Comparing with expression $\mathrm{d} = \mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}t\frac{\partial}{\partial t}$ it follows that

$$\frac{\partial}{\partial x} = v\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial s} + \left(vu + \frac{1}{2}\beta\gamma_1\gamma_2\right)\frac{\partial}{\partial y}. \tag{59}$$

The fact that $\mathrm{d}^2 = 0$ ensures commutativity

$$\left[\frac{\partial}{\partial y}, \frac{\partial}{\partial s}\right] = 0$$

of derivatives with respect to the new variables.

From the above construction, we get

$$\mathrm{d}x = -\left(u + \frac{1}{2}\beta\gamma_1\gamma_2/v\right)\mathrm{d}s + \frac{1}{v}\,\mathrm{d}y.$$

Thus,

$$\left(\frac{1}{v}\right)_s = -\left(u + \frac{1}{2}\beta\gamma_1\gamma_2/v\right)_y. \tag{60}$$

From (59), we find that

$$\frac{\partial x}{\partial y} = \frac{1}{v}, \qquad \frac{\partial x}{\partial s} = -\left(u + \frac{1}{2}\beta\gamma_1\gamma_2/v\right).$$

The first part of equations (42) can also be rewritten as a continuity equation. Define namely

$$P = \frac{m}{v^2} + \beta\frac{\psi_1\psi_2}{2v^3} \tag{61}$$

and

$$\rho = -v - \frac{\beta}{2}P\gamma_1\gamma_2 + \frac{1}{2v}(2\beta\gamma_2\gamma_1 + \beta^2\gamma_2\gamma_{2x} - \beta^2\gamma_1\gamma_{1x}). \tag{62}$$

Then the first part of equations (42) amounts to

$$P_s = \rho_y. \tag{63}$$

Thus, there exists a potential f such that

$$P = f_y, \qquad \rho = f_s.$$

The last two equations of (42) represent fermionic continuity equations and can be used to define fermionic coordinates τ_i, $i = 1, 2$ through e.g.

$$\mathrm{d}\tau_1 = \frac{1}{\sqrt{2}}\psi_1\,\mathrm{d}x + \frac{1}{2}(3u\gamma_1 + 2\beta u\gamma_{2x} + v\gamma_1 + \beta u_x\gamma_2)\,\mathrm{d}t.$$

The coordinate τ_1 satisfies

$$\frac{\partial\tau_1}{\partial x} = \frac{1}{\sqrt{2}}\psi_1, \qquad \frac{\partial\tau_1}{\partial t} = \frac{1}{2}(3u\gamma_1 + 2\beta u\gamma_{2x} + v\gamma_1 + \beta u_x\gamma_2).$$

In hodographic variables the fermionic conservation laws become

$$\begin{aligned}
\left(\frac{\psi_1}{\sqrt{2}v}\right)_s &= \frac{1}{2}\left(u\gamma_1 + v\gamma_1 + \beta u_x\gamma_2 - \frac{\beta}{v}\gamma_1\gamma_2\frac{\psi_1}{\sqrt{2}}\right)_y \\
\left(\frac{\psi_2}{\sqrt{2}v}\right)_s &= \frac{1}{2}\left(u\gamma_2 + v\gamma_2 + \beta u_x\gamma_1 - \frac{\beta}{v}\gamma_1\gamma_2\frac{\psi_2}{\sqrt{2}}\right)_y.
\end{aligned} \tag{64}$$

Thus, after the hodographic transformation all equations of motion of the model are described by continuity equations (60), (63), (64), which all have a common form

$$G_s = H_y.$$

Now, we will investigate the invariance of such continuity equations under the supersymmetry transformations (48). Consider the ϵ_1 part of the supersymmetry transformations (48):

$$\begin{aligned}
\frac{1}{\sqrt{2}}\delta_1\psi_1 &= \epsilon_1\frac{m-v}{\beta}, & \frac{1}{\sqrt{2}}\delta_1\psi_2 &= -\epsilon_1 v_x \\
\delta_1 v &= \epsilon_1\frac{1}{\sqrt{2}}\psi_2, & \delta_1 m &= \epsilon_1\left(\frac{1}{\sqrt{2}}\psi_2 - \frac{\beta}{\sqrt{2}}\psi_{1x}\right).
\end{aligned}$$

For an arbitrary function $F(y, s)$ it holds on the basis of (59) that

$$\delta_1 \frac{\partial}{\partial y} F = \frac{\partial}{\partial y} \delta_1 F - \epsilon_1 \frac{\gamma_2 + \beta \gamma_{1x}}{v} \frac{\partial}{\partial y} F$$

$$\delta_1 \frac{\partial}{\partial s} F = \frac{\partial}{\partial s} \delta_1 F - \frac{\epsilon_1}{2} \left(u\gamma_2 + v\gamma_2 + \beta u_x \gamma_1 - \frac{\beta}{v} \gamma_1 \gamma_2 \frac{\psi_2}{\sqrt{2}} \right) \frac{\partial}{\partial y} F. \tag{65}$$

Let h_1 be a function such that

$$h_{1y} = \frac{\psi_2}{\sqrt{2}v}, \qquad h_{1s} = \frac{1}{2} \left(u\gamma_2 + v\gamma_2 + \beta u_x \gamma_1 - \frac{\beta}{v} \gamma_1 \gamma_2 \frac{\psi_2}{\sqrt{2}} \right)$$

in agreement with (64). Applying δ_1 on a general continuity equation $G_s = H_y$ yields

$$(\delta_1 G - \epsilon_1 h_1 G_y)_s = (\delta_1 H - \epsilon_1 h_1 G_s)_y = (\delta_1 H - \epsilon_1 h_1 H_y)_y,$$

which can be rewritten as a new transformed continuity equation

$$(\widetilde{\delta}_1 G)_s = (\widetilde{\delta}_1 H)_y,$$

where we introduced an extended supersymmetry transformation

$$\widetilde{\delta}_1 = \delta_1 - \epsilon_1 h_1 \frac{\partial}{\partial y}.$$

The action of $\widetilde{\delta}_1$ is explicitly given by

$$\widetilde{\delta}_1 \gamma_1 = \epsilon_1 \frac{u - v}{\beta} - \epsilon_1 h_1 \gamma_{1y}, \qquad \widetilde{\delta}_1 \gamma_2 = -\epsilon_1 u_x - \epsilon_1 h_1 \gamma_{2y}$$

$$\widetilde{\delta}_1 u = \epsilon_1 \gamma_2 - \epsilon_1 h_1 u_y, \qquad \widetilde{\delta}_1 v = \epsilon_1(\gamma_2 + \beta\gamma_{1x}) - \epsilon_1 h_1 v_y. \tag{66}$$

Consider now the ϵ_2 part of the supersymmetry transformations (48) and define h_2 such that

$$h_{2y} = \frac{\psi_1}{\sqrt{2}v}, \qquad h_{2s} = \frac{1}{2} \left(u\gamma_1 + v\gamma_1 + \beta u_x \gamma_2 - \frac{\beta}{v} \gamma_1 \gamma_2 \frac{\psi_1}{\sqrt{2}} \right).$$

In this case, the extended supersymmetry transformation is defined as

$$\widetilde{\delta}_2 = \delta_2 - \epsilon_2 h_2 \frac{\partial}{\partial y}$$

and explicitly given by

$$\widetilde{\delta}_2 \gamma_1 = \epsilon_2 u_x - \epsilon_2 h_2 \gamma_{1y}, \qquad \widetilde{\delta}_2 \gamma_2 = -\epsilon_2 \frac{u - v}{\beta} - \epsilon_2 h_2 \gamma_{2y}$$

$$\widetilde{\delta}_2 u = \epsilon_2 \gamma_1 - \epsilon_2 h_2 u_y, \qquad \widetilde{\delta}_2 v = \epsilon_2(\gamma_1 + \beta\gamma_{2x}) - \epsilon_2 h_2 v_y. \tag{67}$$

The supersymmetry transformation $\widetilde{\delta}_2$ maps the continuity equation $G_s = H_y$ into

$$(\widetilde{\delta}_2 G)_s = (\widetilde{\delta}_2 H)_y.$$

The main point is that the continuity equations $(\widetilde{\delta}_i G)_s = (\widetilde{\delta}_i H)_y$ for $i = 1, 2$ are linear combinations of original continuity equations (60), (63), (64). Thus, the extended supersymmetry transformations $\widetilde{\delta}_i, i = 1, 2$ keep the model invariant and we conclude that

$$\left[\widetilde{\delta}_i, \frac{\partial}{\partial y} \right] = 0, \qquad \left[\widetilde{\delta}_i, \frac{\partial}{\partial s} \right] = 0. \tag{68}$$

To find the action of the extended supersymmetry transformations $\widetilde{\delta}_i, i = 1, 2$ on $h_i, i = 1, 2$, we first calculate

$$\widetilde{\delta}_1 h_{1y} = -\epsilon_1(v + h_1 h_{1y})_y, \qquad \widetilde{\delta}_2 h_{2y} = \epsilon_2(v - h_1 h_{1y})_y$$

and

$$\widetilde{\delta}_2 h_{1y} = -\epsilon_2 \left(\frac{m-v}{\beta v} + (h_2 h_{1y})_y \right), \qquad \widetilde{\delta}_1 h_{2y} = \epsilon_1 \left(\frac{m-v}{\beta v} - (h_1 h_{2y})_y \right).$$

Thus,

$$\widetilde{\delta}_1 h_1 = -\epsilon_1 (v + h_1 h_{1y} + c_{11}), \qquad \widetilde{\delta}_2 h_2 = \epsilon_2 (v - h_1 h_{1y} + c_{22})$$

and

$$\widetilde{\delta}_2 h_1 = -\epsilon_2 \left(\int \mathrm{d}y \frac{m-v}{\beta v} + h_2 h_{1y} + c_{21} \right), \qquad \widetilde{\delta}_1 h_{2y} = \epsilon_1 \left(\int \mathrm{d}y \frac{m-v}{\beta v} - h_1 h_{2y} + c_{12} \right),$$

where c_{ij}, i, $j = 1, 2$ are integration constants.

Despite the presence of non-local terms in the above extended supersymmetry transformations, one finds that their repeated application yields

$$\widetilde{\delta}_1' \widetilde{\delta}_1 X = \pm \epsilon_1 \epsilon_1' c_{11} X_y, \qquad \widetilde{\delta}_2' \widetilde{\delta}_2 X = \pm \epsilon_2 \epsilon_2' c_{22} X_y,$$

where X stands for u, v, γ_1, γ_2. Similarly,

$$(\widetilde{\delta}_2 \widetilde{\delta}_1 - \widetilde{\delta}_1 \widetilde{\delta}_2) X = \epsilon_1 \epsilon_2 (c_{21} - c_{12}) X_y.$$

Choosing the integration constants as $c_{11} = c_{22} = 1$ and $c_{21} = c_{12}$, we reproduce the $N = 2$ algebra of the supersymmetry transformations given in equation (48).

6. Outlook

In this paper, we have described a general formalism which associates a bi-Hamiltonian structure with the $N = 2$ superconformal Poisson algebra by taking the advantage of the existence of the central extension of the original algebra. We have shown how this deformation uniquely determines a deformed Sobolev-type inner product for the co-adjoint orbit formalism through the construction of the momentum operator for the first bracket. The momentum operator is invariant under the supersymmetry transformation and induces via the Lenard relations a chain of conserved Hamiltonians determining the $N = 2$ supersymmetric Camassa–Holm hierarchy.

All the steps of the above construction are general in nature and the formalism extends easily to other similar formalisms. In a forthcoming publication [24] we will apply the formalism to the $N = 1$ superconformal algebra and also provide formal proofs for the observations made here.

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